Discontinuous Galerkin for diffusion problems: historical overview

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SECOND-ORDER OPERATORS IN DG

Motivation

- Discretization of selft-adjoint operators in convection dominated problems: Navier-Stokes, convection-diffusion equation, Euler equations with artificial viscosity for shock capturing,…

$$\mathbf{U}_t + \nabla \cdot \mathbf{F}(\mathbf{U}) - \nabla \cdot (\varepsilon \nabla \mathbf{U}) = 0$$

How to treat the self-adjoint operator with a DG formulation?

Interior Penalty Method (IPM)
Local Discontinuous Galerkin (LDG)
…

Interior Penalty Method (IPM)

Douglas N. Arnold (1982)

- Model problem over computational domain

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \end{cases}$$

- Model problem over "BROKEN" computational domain

$$\begin{cases} -\Delta u = f & \text{in } \Omega_e, \quad \boxed{\text{for } e = 1, \dots, n_{\texttt{el}}} \\ u = g & \text{on } \Gamma_D, \\ [\![u\mathbf{n}]\!] = 0 & \text{on } \Gamma, \\ [\![\nabla u \cdot \mathbf{n}]\!] = 0 & \text{on } \Gamma, \end{cases}$$

IMPOSE CONTINUITY OF SOLUTION AND FLUXES

Definitions

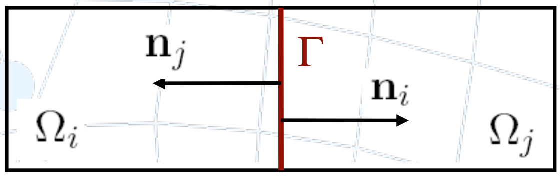
- Computational domain: $\Omega \subset \mathbb{R}^{\texttt{n}_{\texttt{sd}}}$
- With boundary $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$
- Ω is partitioned in $\texttt{n}_{\texttt{el}}$ disjoint subdomains Ω_i s.t.

$$\overline{\Omega} = \bigcup_{i=1}^{\texttt{n}_{\texttt{el}}} \overline{\Omega}_i, \qquad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j$$

- with boundaries $\partial\Omega_i$, which define an internal interface Γ

$$\Gamma := \left[\bigcup_{i=1}^{\texttt{n}_{\texttt{el}}} \partial\Omega_i \right] \setminus \partial\Omega$$

Notation



[Montlaur, A., Fernández-Méndez, S., Huerta, A. IJNMF'08]

$$[\![u\mathbf{n}]\!] = \begin{cases} u_i\mathbf{n}_i + u_j\mathbf{n}_j & \text{on } \Gamma \\ u\mathbf{n} & \text{on } \partial\Omega \end{cases} \qquad \text{for scalars}$$

$$[\![\boldsymbol{\sigma}\cdot\mathbf{n}]\!] = \begin{cases} \boldsymbol{\sigma}_i\cdot\mathbf{n}_i + \boldsymbol{\sigma}_j\cdot\mathbf{n}_j & \text{on } \Gamma \\ \boldsymbol{\sigma}\cdot\mathbf{n} & \text{on } \partial\Omega \end{cases} \qquad \text{for vectors}$$

$$\{u\} = \begin{cases} \frac{1}{2}(u_i + u_j) & \text{on } \Gamma \\ u & \text{on } \partial\Omega \end{cases} \qquad \text{for scalars}$$

$$\{\boldsymbol{\sigma}\} = \begin{cases} \frac{1}{2}(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j) & \text{on } \Gamma \\ \boldsymbol{\sigma} & \text{on } \partial\Omega \end{cases} \qquad \text{for vectors}$$

Interior Penalty Method (IPM)

Douglas N. Arnold (1982)

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IMPOSE CONTINUITY OF SOLUTION AND FLUXES

What happens with Stokes?

- The strong form
$$\begin{cases} -\boldsymbol{\nabla}\cdot(\nu\boldsymbol{\nabla}\boldsymbol{u})+\boldsymbol{\nabla}p=\boldsymbol{s} & \text{in } \Omega,\\ \boldsymbol{\nabla}\cdot\boldsymbol{u}=0 & \text{in } \Omega,\\ \boldsymbol{u}=\boldsymbol{u}_D & \text{on } \Gamma_D,\\ \nu(\boldsymbol{n}\cdot\boldsymbol{\nabla})\boldsymbol{u}-p\boldsymbol{n}=\boldsymbol{t} & \text{on } \Gamma_N,\end{cases}$$

- Model problem over "BROKEN" computational domain
$$\begin{cases} -\boldsymbol{\nabla}\cdot(\nu\boldsymbol{\nabla}\boldsymbol{u})+\boldsymbol{\nabla}p=\boldsymbol{s} & \text{in } \Omega_e,\\ \boldsymbol{\nabla}\cdot\boldsymbol{u}=0 & \text{in } \Omega_e, \quad \boxed{\text{for } e=1,\dots,n_{\mathtt{el}}}\\ \boldsymbol{u}=\boldsymbol{u}_D & \text{on } \Gamma_D\cap\partial\Omega_e,\\ (\nu\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{n}-p\boldsymbol{n}=\boldsymbol{t} & \text{on } \Gamma_N\cap\partial\Omega_e,\\ [\![\boldsymbol{u}\otimes\boldsymbol{n}]\!]=\boldsymbol{0} & \text{on } \Gamma,\\ [\![(\nu\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{n}-p\boldsymbol{n}]\!]=\boldsymbol{0} & \text{on } \Gamma,\end{cases}$$

IMPOSE CONTINUITY OF SOLUTION AND FLUXES

Interior Penalty Method (IPM)

Douglas N. Arnold (1982)

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$$\begin{cases} -\Delta u=f & \text{in } \Omega,\\ u=g & \text{on } \Gamma_D,\end{cases}$$

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IMPOSE CONTINUITY OF SOLUTION AND FLUXES

Interior Penalty Method (IPM)

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$$\begin{cases} -\Delta u = f & \text{in } \Omega_e, \quad \text{for } e = 1, \ldots, n_{\mathtt{el}} \\ u = g & \text{on } \Gamma_D, \\ [\![u\mathbf{n}]\!] = 0 & \text{on } \Gamma, \\ [\![\nabla u \cdot \mathbf{n}]\!] = 0 & \text{on } \Gamma, \end{cases}$$

- Weak formulation in a generic element

$$\int_{\Omega_e} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \; d\Omega - \int_{\partial\Omega_e} (\boldsymbol{\nabla} u \cdot \mathbf{n}) \; v \; d\Gamma = \int_{\Omega_e} f \; v \; d\Omega$$

where **n** is the unitary outward normal to $\partial\Omega_e$

IPM. Weak formulation

- Adding over elements

$$\boxed{\int_{\Omega} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \; d\Omega - \boxed{\sum_e \int_{\partial\Omega_e} (\boldsymbol{\nabla} u \cdot \mathbf{n}) \; v \; d\Gamma} = \int_{\Omega} f \; v \; d\Omega}$$

Useful identity (I)

$$\boxed{\sum_{e=1}^{\mathtt{n_{el}}} \int_{\partial\Omega_e} \alpha \boldsymbol{w} \cdot \boldsymbol{n} d\Gamma = \int_{\Gamma} \Big([\![\alpha \boldsymbol{n}]\!] \cdot \{\boldsymbol{w}\} + \{\alpha\} [\![\boldsymbol{w} \cdot \boldsymbol{n}]\!] \Big) d\Gamma + \int_{\partial\Omega} \alpha \boldsymbol{w} \cdot \boldsymbol{n} d\Gamma}$$

Γ or Γ_{int} is the union of all interior edges/faces

$\partial\Omega$ is the union of all exteriors edges/faces, which can be split in Dirichlet, Γ_D, and Neumann, Γ_N, boundaries

Useful identity

- Useful identity:

$$\sum_{e=1}^{\mathtt{n_{el}}} \int_{\partial\Omega_e} \alpha \boldsymbol{w}\cdot\boldsymbol{n}\, d\Gamma = \int_{\Gamma} \Big([\![\alpha \boldsymbol{n}]\!]\cdot\{\boldsymbol{w}\} + \{\alpha\}[\![\boldsymbol{w}\cdot\boldsymbol{n}]\!] \Big) d\Gamma + \int_{\partial\Omega} \alpha \boldsymbol{w}\cdot\boldsymbol{n}\, d\Gamma$$

- Assume Ω_i and Ω_j are adjacent elements, for that edge/face

$$\sum_{e=i,j} \int_{\partial\Omega_e} \alpha \boldsymbol{w}\cdot\boldsymbol{n}\, d\Gamma = \int_{\partial\Omega_i\cap\partial\Omega_j} \Big(\alpha_i \boldsymbol{w}_i\cdot\boldsymbol{n}_i + \alpha_j \boldsymbol{w}_j\cdot\boldsymbol{n}_j \Big) d\Gamma$$

$$\boxed{\overset{?}{=}} \int_{\partial\Omega_i\cap\partial\Omega_j} \Big([\![\alpha \boldsymbol{n}]\!]\cdot\{\boldsymbol{w}\} + \{\alpha\}[\![\boldsymbol{w}\cdot\boldsymbol{n}]\!] \Big) d\Gamma$$

$$\int_{\partial\Omega_i\cap\partial\Omega_j} \Big([\![\alpha \boldsymbol{n}]\!]\cdot\{\boldsymbol{w}\} + \{\alpha\}[\![\boldsymbol{w}\cdot\boldsymbol{n}]\!] \Big) d\Gamma =$$

$$\int_{\partial\Omega_i\cap\partial\Omega_j} \Big(\tfrac{1}{2}(\alpha_i \boldsymbol{n}_i + \alpha_j \boldsymbol{n}_j)\cdot(\boldsymbol{w}_i + \boldsymbol{w}_j) + \tfrac{1}{2}(\alpha_i + \alpha_j)(\boldsymbol{w}_i\cdot\boldsymbol{n}_i + \boldsymbol{w}_j\cdot\boldsymbol{n}_j) \Big) d\Gamma$$

$$= \int_{\partial\Omega_i\cap\partial\Omega_j} \Big(\alpha_i \boldsymbol{w}_i\cdot\boldsymbol{n}_i + \alpha_j \boldsymbol{w}_j\cdot\boldsymbol{n}_j \Big) d\Gamma +$$

$$\frac{1}{2}\int_{\partial\Omega_i\cap\partial\Omega_j} \Big(\alpha_i \boldsymbol{w}_j\cdot\boldsymbol{n}_i + \alpha_j \boldsymbol{w}_i\cdot\boldsymbol{n}_j + \alpha_i \boldsymbol{w}_j\cdot\boldsymbol{n}_j + \alpha_j \boldsymbol{w}_i\cdot\boldsymbol{n}_i \Big) d\Gamma$$

$$= \sum_{e=i,j} \int_{\partial\Omega_e} \alpha \boldsymbol{w}\cdot\boldsymbol{n}\, d\Gamma$$

IPM. Weak formulation

- Adding over elements

$$\boxed{\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \sum_e \int_{\partial\Omega_e} (\nabla u \cdot \mathbf{n}) \, v \, d\Gamma = \int_\Omega f \, v \, d\Omega}$$

Useful identity (I)

$$\sum_{e=1}^{\mathtt{n_{el}}} \int_{\partial\Omega_e} \alpha \boldsymbol{w} \cdot \boldsymbol{n} d\Gamma = \int_\Gamma \Big([\![\alpha \boldsymbol{n}]\!] \cdot \{\boldsymbol{w}\} + \{\alpha\} [\![\boldsymbol{w} \cdot \boldsymbol{n}]\!] \Big) d\Gamma + \int_{\partial\Omega} \alpha \boldsymbol{w} \cdot \boldsymbol{n} d\Gamma$$

$$\boxed{[\![\nabla u \cdot \mathbf{n}]\!] = 0}$$

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \boxed{\int_{\Gamma \cup \Gamma_D} ([\![v\mathbf{n}]\!] \cdot \{\nabla u\}} + \cancel{\{v\}[\![\nabla u \cdot \mathbf{n}]\!]}) \, d\Gamma = \int_\Omega f v \, d\Omega$$

non-symmetric

Recall $\partial\Omega = \Gamma_D$

IPM. Weak formulation

- Adding terms to obtain a symmetric bilinear form

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma \cup \Gamma_D} \Big([\![v\mathbf{n}]\!] \cdot \{\nabla u\} \qquad\qquad \Big) \, d\Gamma$$

$$= \int_\Omega f v \, d\Omega$$

IPM. Weak formulation

- Adding terms to obtain a symmetric bilinear form

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma \cup \Gamma_D} \Big([\![v\mathbf{n}]\!] \cdot \{\nabla u\} + [\![u\mathbf{n}]\!] \cdot \{\nabla v\} \Big) \, d\Gamma$$
$$= \int_\Omega f v \, d\Omega - 0 - \int_{\Gamma_D} g \nabla v \cdot \mathbf{n} \, d\Gamma$$

IPM. Weak formulation

- Adding terms to obtain a symmetric bilinear form

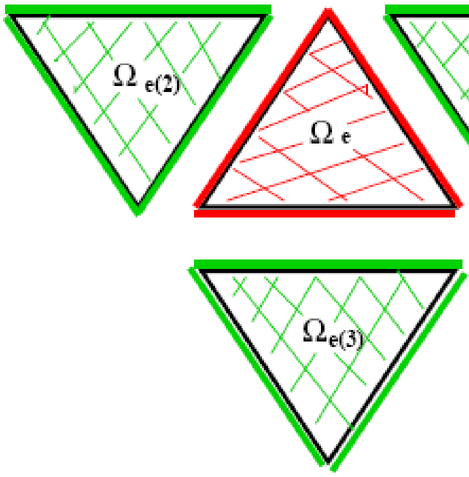
$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma \cup \Gamma_D} \Big([\![v\mathbf{n}]\!] \cdot \{\nabla u\} + [\![u\mathbf{n}]\!] \cdot \{\nabla v\} \Big) \, d\Gamma$$
$$= \int_\Omega f v \, d\Omega - 0 - \int_{\Gamma_D} g \nabla v \cdot \mathbf{n} \, d\Gamma$$

- Now it is symmetric, but maybe not coercive. Add terms

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma \cup \Gamma_D} \Big([\![v\mathbf{n}]\!] \cdot \{\nabla u\} + [\![u\mathbf{n}]\!] \cdot \{\nabla v\} \Big) \, d\Gamma + \int_{\Gamma \cup \Gamma_D} \beta [\![u\mathbf{n}]\!] \cdot [\![v\mathbf{n}]\!] \, d\Gamma$$
$$= \int_\Omega f v \, d\Omega - 0 - \int_{\Gamma_D} g \nabla v \cdot \mathbf{n} \, d\Gamma + 0 + \int_{\Gamma_D} \beta \, g v \, d\Gamma$$

- The bilinear form is coercive for β large enough.
  $\boxed{\beta = \alpha h^{-1}}$ ensures optimal convergence (consistent penalty).
  constant typical of Nitsche BC

IPM. Stencil 2D



Only neighboring elements adjacent to Ω_e are used

But ∇u must be calculated on $\partial\Omega_e$
(thus, dependence on all neighboring nodes)

IPM convergence

- Using polynomials of degree p the following optimal rates of convergence are demonstrated

Norm	Order of convergence
$\mathcal{L}^2$	p+1
$\mathcal{H}^1$	p

- If the penalty parameter is not defined as $\beta = \alpha h^{-1}$ the optimal rate of convergence can be degraded.

Local Discontinuous Galerkin (LDG)

Cockburn and Shu (1998)

- Model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \end{cases}$$

- Mixed formulation (system of first-order PDEs):

$$\begin{cases} \boldsymbol{\sigma} - \boldsymbol{\nabla} u = \mathbf{0} & \text{in } \Omega, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \end{cases}$$

- Mixed formulation and broken computational domain

$$\begin{cases} \boldsymbol{\sigma} - \boldsymbol{\nabla} u = \mathbf{0} & \text{in } \Omega_e, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f & \text{in } \Omega_e, \\ \boxed{\begin{aligned} [\![u\, \boldsymbol{n}]\!] &= \mathbf{0} \quad \text{on } \Gamma, \\ [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] &= 0 \quad \text{on } \Gamma, \end{aligned}} \\ u = g & \text{on } \partial\Omega := \Gamma_D. \end{cases}$$

for $e = 1, \dots, n_{\texttt{el}}$

IMPOSE CONTINUITY OF SOLUTION AND FLUXES

LDG. Weak formulation

$$\begin{cases} \boldsymbol{\sigma} - \boldsymbol{\nabla} u = \mathbf{0} & \text{in } \Omega_e, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f & \text{in } \Omega_e, \\ u = g & \text{on } \partial\Omega. \end{cases} \qquad \begin{cases} [\![u\, \boldsymbol{n}]\!] = \mathbf{0} & \text{on } \Gamma, \\ [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] = 0 & \text{on } \Gamma, \end{cases}$$

- Weak formulation on Ω_e

$$\int_{\Omega_e} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, d\Omega + \int_{\Omega_e} u \, \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \, d\Omega - \int_{\partial\Omega_e} u \, \mathbf{n} \cdot \boldsymbol{\tau} \, d\Gamma = 0$$

$$\int_{\Omega_e} \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v \, d\Omega - \int_{\partial\Omega_e} \boldsymbol{\sigma} \cdot \mathbf{n} \, v \, d\Gamma = \int_{\Omega_e} f \, v \, d\Omega$$

- Numerical fluxes

$$\int_{\Omega_e} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, d\Omega + \int_{\Omega_e} u \, \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \, d\Omega - \int_{\partial\Omega_e} \boxed{\hat{u}} \, \mathbf{n} \cdot \boldsymbol{\tau} \, d\Gamma = 0$$

$$\int_{\Omega_e} \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v \, d\Omega - \int_{\partial\Omega_e} \boxed{\hat{\boldsymbol{\sigma}}} \cdot \mathbf{n} \, v \, d\Gamma = \int_{\Omega_e} f \, v \, d\Omega$$

LDG. Numerical fluxes

- The numerical fluxes are defined as

$$\boxed{\begin{aligned} \hat{u} &:= \{u\} + \mathbf{C_{12}} \cdot [\![u\mathbf{n}]\!] \\ \hat{\boldsymbol{\sigma}} &:= \{\boldsymbol{\sigma}\} - \mathbf{C_{12}} [\![\boldsymbol{\sigma} \cdot \mathbf{n}]\!] - C_{11} [\![u\mathbf{n}]\!] \end{aligned}}$$

with $\boldsymbol{C}_{12} = \frac{1}{2}(S_{ij}\mathbf{n}_i + S_{ji}\mathbf{n}_j)$

and a switch such that $S_{ij} + S_{ji} = 1$

- Some properties:
  - The u-flux does not depend on σ
  - Consistency $\hat{u}(u) = u_{|\Gamma}$, $\hat{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, u) = \boldsymbol{\sigma}_{|\Gamma}$
  - Conservation $[\![\hat{u}\mathbf{n}]\!] = \hat{u}_i \mathbf{n}_i + \hat{u}_j \mathbf{n}_j = \mathbf{0}$, $[\![\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}]\!] = \hat{\boldsymbol{\sigma}}_i \cdot \mathbf{n}_i + \hat{\boldsymbol{\sigma}}_j \cdot \mathbf{n}_j = 0$ $\qquad \boxed{\begin{cases} [\![u\, \boldsymbol{n}]\!] = \mathbf{0} \\ [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] = 0 \end{cases}}$

LOCAL DG

LDG summary

LOCAL $$\int_{\Omega_e} \boxed{\boldsymbol{\sigma}} \cdot \boldsymbol{\tau} \, d\Omega + \int_{\Omega_e} u \, \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \, d\Omega - \int_{\partial\Omega_e} \boxed{\hat{u}} \, \mathbf{n} \cdot \boldsymbol{\tau} \, d\Gamma = 0$$

GLOBAL $$\int_{\Omega_e} \boxed{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} v \, d\Omega - \int_{\partial\Omega_e} \boxed{\hat{\boldsymbol{\sigma}}} \cdot \mathbf{n} \, v \, d\Gamma = \int_{\Omega_e} f \, v \, d\Omega$$

with $$\boxed{\hat{u}} := \{u\} + \mathbf{C}_{12} \cdot [\![u\mathbf{n}]\!]$$

$$\boxed{\hat{\boldsymbol{\sigma}}} := \{\boxed{\boldsymbol{\sigma}}\} - \mathbf{C}_{12} [\![\boxed{\boldsymbol{\sigma}} \cdot \mathbf{n}]\!] - C_{11} [\![u\mathbf{n}]\!]$$

- **σ is isolated from LOCAL and replaced in GLOBAL**

This can be done after discretization…
or in the weak form using the so-called “lifting operators”

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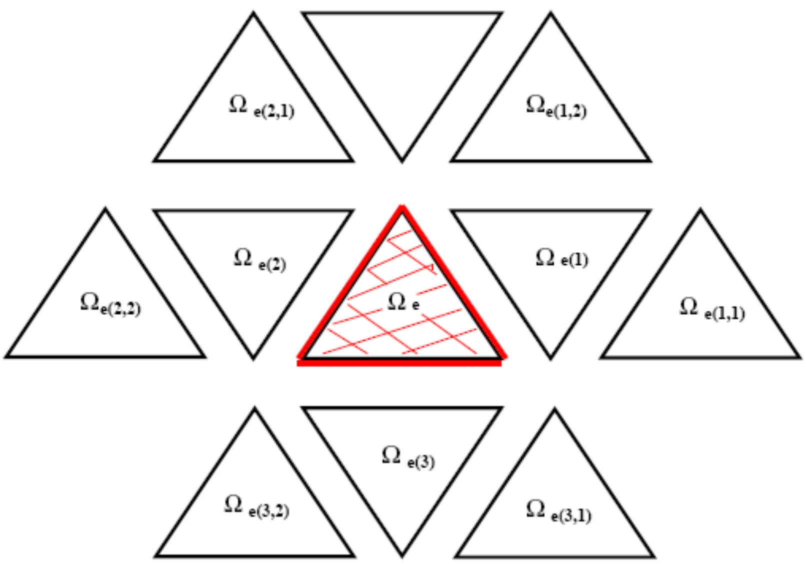
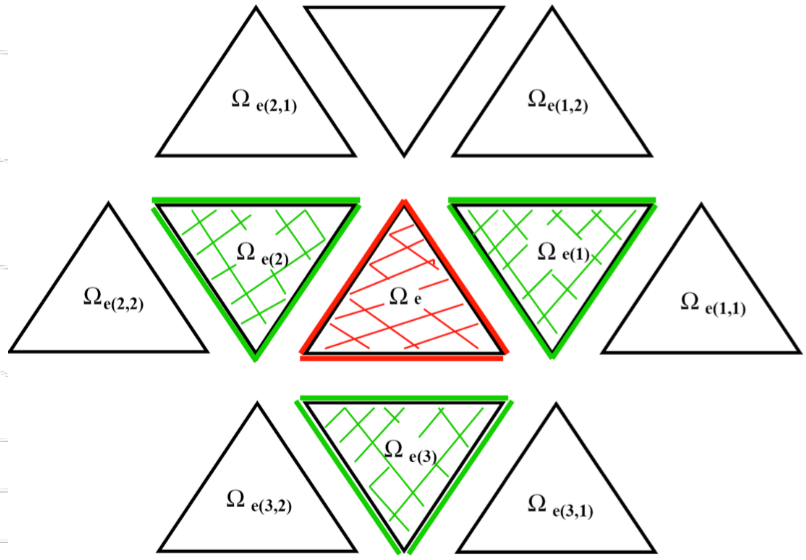
LDG Stencil 2D



Global equation at element Ω_e uses traces $\hat{\boldsymbol{\sigma}}$ of neighbors:

$$\boxed{\int_{\Omega_e} \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v \, d\Omega - \int_{\partial\Omega_e} \hat{\boldsymbol{\sigma}} \cdot \mathbf{n} \, v \, d\Gamma = \int_{\Omega_e} f \, v \, d\Omega}$$

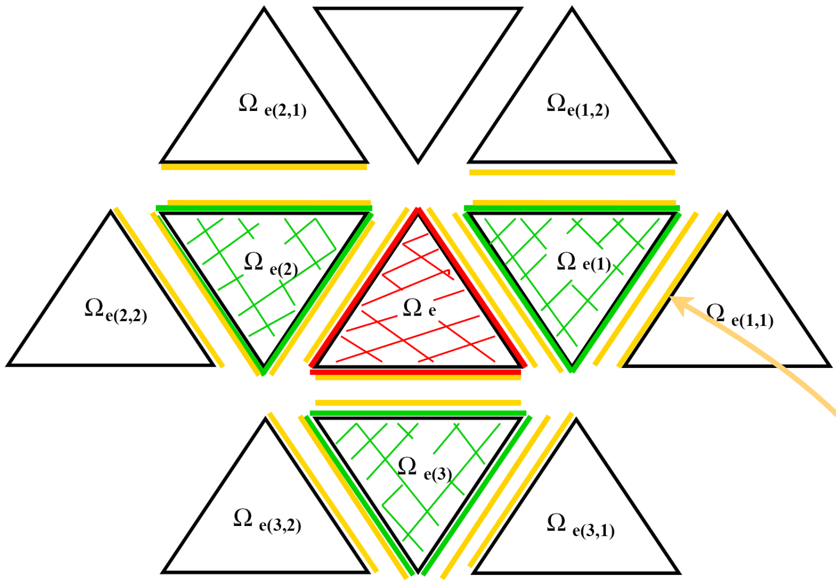
$$\hat{\boldsymbol{\sigma}} := \{\boldsymbol{\sigma}\} - \mathbf{C_{12}}[\![\boldsymbol{\sigma} \cdot \mathbf{n}]\!] - C_{11}[\![u\mathbf{n}]\!]$$



But $\hat{\boldsymbol{\sigma}}$ of neighbors requires solving local equation on red and green elements

$$\boxed{\int_{\Omega_e} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, d\Omega + \int_{\Omega_e} u \, \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \, d\Omega - \int_{\partial\Omega_e} \hat{u} \, \mathbf{n} \cdot \boldsymbol{\tau} \, d\Gamma = 0}$$

$$\hat{u} := \{u\} + \mathbf{C_{12}} \cdot [\![u\mathbf{n}]\!]$$



Local equation uses traces of primal variable (no derivatives)

$$\boxed{\int_{\Omega_e} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, d\Omega + \int_{\Omega_e} u \, \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \, d\Omega - \int_{\partial\Omega_e} \hat{u} \, \mathbf{n} \cdot \boldsymbol{\tau} \, d\Gamma = 0}$$

Are LDG and IPM alike?

- Integrate by parts **LOCAL** equation

$$\int_{\Omega_e} \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,d\Omega - \int_{\Omega_e} \boldsymbol{\nabla} u\cdot\boldsymbol{\tau}\,d\Omega + \int_{\partial\Omega_e} (u-\widehat{u})\boldsymbol{\tau}\cdot\mathbf{n}\,d\Gamma = 0$$

- Sum over elements, and apply identity (I) to the previous equation and the **GLOBAL** equation. **LDG** is rewritten as

LOCAL

$$\int_{\Omega} \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,d\Omega - \int_{\Omega} \boldsymbol{\nabla} u\cdot\boldsymbol{\tau}\,d\Omega + \int_{\Gamma\cup\Gamma_D} [\![u\mathbf{n}]\!]\cdot\{\boldsymbol{\tau}\}\,d\Gamma + \int_{\Gamma} \{u-\widehat{u}\}[\![\tau\cdot\mathbf{n}]\!]\,d\Gamma = 0$$

GLOBAL $\int_{\Omega} \boldsymbol{\sigma}\cdot\boldsymbol{\nabla} v\,d\Omega = \int_{\Omega} f v\,d\Omega + \int_{\Gamma\cup\Gamma_D} \{\widehat{\boldsymbol{\sigma}}\}\cdot[\![v\mathbf{n}]\!]\,d\Gamma$

- **Recall:** determine **σ** from **LOCAL** and replace in **GLOBAL** to get an equation with only u

Lifting operators

$$\int_{\Omega} \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,d\Omega - \int_{\Omega} \boldsymbol{\nabla} u\cdot\boldsymbol{\tau}\,d\Omega + \int_{\Gamma\cup\Gamma_D} [\![u\mathbf{n}]\!]\cdot\{\boldsymbol{\tau}\}\,d\Gamma + \int_{\Gamma} \{u-\widehat{u}\}[\![\tau\cdot\mathbf{n}]\!]\,d\Gamma = 0$$

LOCAL

LOCAL in strong form

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\nabla} u + r([\![u\mathbf{n}]\!]) + \ell(\{u-\widehat{u}\})}$$

[Remark: here u and σ denote the LDG solution, not the analytical solution]

The *lifting operators* ℓ and r are defined as

$$\int_{\Omega} r(\boldsymbol{\phi})\cdot\boldsymbol{\tau}\,d\Omega = -\int_{\Gamma\cup\Gamma_D} \boldsymbol{\phi}\cdot\{\boldsymbol{\tau}\}\,d\Gamma \qquad \forall\boldsymbol{\tau}$$

$$\int_{\Omega} \ell(q)\cdot\boldsymbol{\tau}\,d\Omega = -\int_{\Gamma} q[\![\boldsymbol{\tau}\cdot\mathbf{n}]\!]\,d\Gamma \qquad \forall\boldsymbol{\tau}$$

Now **LOCAL** in strong form can be replaced in **GLOBAL**…

$$\int_{\Omega} \boldsymbol{\sigma}\cdot\boldsymbol{\nabla} v\,d\Omega = \int_{\Omega} f v\,d\Omega + \int_{\Gamma\cup\Gamma_D} \{\widehat{\boldsymbol{\sigma}}\}\cdot[\![v\mathbf{n}]\!]\,d\Gamma \qquad \widehat{\boldsymbol{\sigma}} := \{\boldsymbol{\sigma}\} - \mathbf{C}_{12}[\![\boldsymbol{\sigma}\cdot\mathbf{n}]\!] - C_{11}[\![u\mathbf{n}]\!]$$

LDG primal form

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma \cup \Gamma_D} [\![u\mathbf{n}]\!] \cdot \{\nabla v\} \, d\Gamma$$

$$- \int_{\Gamma \cup \Gamma_D} \{\nabla u\} \cdot [\![v\mathbf{n}]\!] \, d\Gamma + \int_{\Gamma \cup \Gamma_D} C_{11} [\![u\mathbf{n}]\!] \cdot [\![v\mathbf{n}]\!] \, d\Gamma$$

$$- \int_\Gamma (\boldsymbol{C}_{12} \cdot [\![u\mathbf{n}]\!] [\![\nabla v \cdot \mathbf{n}]\!] + [\![\nabla u \cdot \mathbf{n}]\!] \boldsymbol{C}_{12} \cdot [\![v\mathbf{n}]\!]) \, d\Gamma$$

$$+ \int_\Omega (\, r([\![u\mathbf{n}]\!]) + \ell(\boldsymbol{C}_{12} \cdot [\![u\mathbf{n}]\!])) \cdot (\, r([\![v\mathbf{n}]\!]) + \ell(\boldsymbol{C}_{12} \cdot [\![v\mathbf{n}]\!])) \, d\Gamma$$

$$= \int_\Omega f v \, d\Omega + \int_{\Gamma_D} C_{11} g v \, d\Gamma - \int_{\Gamma_D} g \nabla v \cdot \mathbf{n}$$

$$- \int_{\Gamma_D} g \, (\, r([\![v\mathbf{n}]\!]) + \ell(\boldsymbol{C}_{12} \cdot [\![v\mathbf{n}]\!])) \, d\Gamma$$

LDG weak form Ξ IPM weak form + extra terms

LDG convergence

- Using polynomials of degree p the following optimal rates of convergence are demonstrated

Norm	Order of convergence
$\mathcal{L}^2$	p+1
$\mathcal{H}^1$	p

- Recall: $\hat{u} := \{u\} + \mathbf{C_{12}} \cdot [\![u\mathbf{n}]\!]$

  $\hat{\boldsymbol{\sigma}} := \{\boldsymbol{\sigma}\} - \mathbf{C_{12}}[\![\boldsymbol{\sigma} \cdot \mathbf{n}]\!] - C_{11}[\![u\mathbf{n}]\!]$

- The optimal order of convergence in the $\mathcal{L}^2$ norm is obtained when the parameter C_{11} is mesh-dependent (C_{11} must be h^{-1} like the penalty parameter of the IPM).

  If C_{11} is constant the order is not optimal (p+1/2).

Compact Discontinuous Galerkin (CDG)

- LDG stencil is larger than IPM stencil: lost of compactness due to the lifting operators
- **CDG** [Peraire & Persson SISC'08]: modify liftings to keep compactness

Instead of $\boxed{\boldsymbol{\sigma} = \boldsymbol{\nabla} u + r([\![u\mathbf{n}]\!]) + \ell(\{u - \widehat{u}\})}$, solution of the LOCAL problem, CDG considers for each face i

$$\boxed{\boldsymbol{\sigma}^i = \boldsymbol{\nabla} u + r^i([\![u\mathbf{n}]\!]) + \ell^i(\{u - \widehat{u}\})}$$

with the modified liftings

$$\int_\Omega r^i(\boldsymbol{\phi}) \cdot \boldsymbol{\tau}\, d\Omega = -\int_{\Gamma_i} \boldsymbol{\phi} \cdot \{\boldsymbol{\tau}\}\, d\Gamma \qquad \forall \boldsymbol{\tau}$$

$$\int_\Omega \ell^i(q) \cdot \boldsymbol{\tau}\, d\Omega = -\int_{\Gamma_i} q[\![\boldsymbol{\tau} \cdot \mathbf{n}]\!]\, d\Gamma \qquad \forall \boldsymbol{\tau}$$

CGD weak form similar to LDG but compact scheme

Comparison IPM and CDG

IPM	CDG
compact methods (relative to LDG)	
optimal convergence rates	
similar accuracy	
straight-forward rationale and implementation	non trivial implementation and extra computational cost of lifting operators
necessary tuning of penalty parameter C_{11}	less sensitive to the selection of C_{11} parameter

[Montaur, Fernández-Méndez, Peraire, AH IJNMF'09]

DG unified analysis for self-adjoint operators

[Arnold, Brezzi, Cockburn and Marini, SINUM'02]

Method	$\widehat{u}_K$	$\widehat{\sigma}_K$
Bassi–Rebay [9]	$\{u_h\}$	$\{\sigma_h\}$
Brezzi et al. [18]	$\{u_h\}$	$\{\sigma_h\} - \alpha_{\mathrm{r}}([\![u_h]\!])$
LDG [35]	$\{u_h\} - \beta \cdot [\![u_h]\!]$	$\{\sigma_h\} + \beta[\![\sigma_h]\!] - \alpha_{\mathrm{j}}([\![u_h]\!])$
IP [43]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_{\mathrm{j}}([\![u_h]\!])$
Bassi et al. [10]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_{\mathrm{r}}([\![u_h]\!])$
Baumann–Oden [12]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\}$
NIPG [53]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\} - \alpha_{\mathrm{j}}([\![u_h]\!])$
Babuška–Zlámal [6]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_{\mathrm{j}}([\![u_h]\!])$
Brezzi et al. [19]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_{\mathrm{r}}([\![u_h]\!])$

High vs. Low-order

Do we need high-order?
Literature conclusions are non-conclusive:

- [Vos, Sherwin & Kirby, JCP'10] :
  "for a low error level of 10% a reasonably coarse mesh with a sixth-order spectral/hp expansions minimised the run-time"
- [Löhner, IJNMF'11+'13] :
  "The comparison of error and work estimates shows that for relative accuracy in the 0.1% range, which is one order below the typical accuracy of engineering interest (1% range), linear elements may outperform all high-order elements."
- … [AH, A. Aleksandar, X. Roca, J. Peraire, IJNME'13]

Computational cost estimate

- Compare for different:
  - Galerkin methods: CG, CG(NSC), CDG and HDG
  - Element types: simplices/paralellotopes in 2D/3D
  - Approximation orders *p* (low versus high)
- How to evaluate computational cost:
  - Asymptotic estimates: major uncertainties
  - Cost indicators (number of: elements, DOF, non-zeros per row, non-zeros): not enough information
  - **Operation count:** cost of local (element-by-element) and global operations ... (memory operations)
- To compute cost estimates evaluate FLOPS for
  - Creating element and face matrices
  - Solving the local problem

  } Parallelizable
  - Solving the global problem

Computational cost estimate

- Major hypothesis:
  - Structured uniform mesh having a number of boundary faces negligible compared with the number of interior ones,
  - Smooth solution (bounded solution & bounded derivatives) and such that the approximation error is controlled by the interpolation one
- Compare computational cost to achieve the same level of accuracy
- Estimate ratio between low and high order elements for a given approximation error

$$\boxed{n_{e,1}/n_{e,p} = 2^{-d/2}\epsilon^{-(d/2)(p-1)/(p+1)}\left((p+1)!\right)^{d/(p+1)} \geq 1,}$$

Table I. Expressions for `ndof` for different methods.

		ndof
Triangles	CG	$n_{e,p}(3\mathtt{ndof}_{f,p} - 5)/2$
	HDG	$3n_{e,p}\,\mathtt{ndof}_{f,p}/2$
	CG(NSC)	$n_{e,p}(2\mathtt{ndof}_{e,p} - 3\mathtt{ndof}_{f,p} + 1)/2$
	CDG	$n_{e,p}\,\mathtt{ndof}_{e,p}$
Quads	CG	$n_{e,p}(2\mathtt{ndof}_{f,p} - 3)$
	HDG	$2n_{e,p}\,\mathtt{ndof}_{f,p}$
	CG(NSC)	$n_{e,p}(\mathtt{ndof}_{e,p} - 2\mathtt{ndof}_{f,p} + 1)$
	CDG	$n_{e,p}\,\mathtt{ndof}_{e,p}$
Tets	CG	$n_{e,p}(12\mathtt{ndof}_{f,p} - 29\mathtt{ndof}_{g,p} + 23)/6$
	HDG	$2n_{e,p}\,\mathtt{ndof}_{f,p}$
	CG(NSC)	$n_{e,p}(6\mathtt{ndof}_{e,p} - 12\mathtt{ndof}_{f,p} + 7\mathtt{ndof}_{g,p} - 1)/6$
	CDG	$n_{e,p}\,\mathtt{ndof}_{e,p}$
Hexes	CG	$n_{e,p}(3\mathtt{ndof}_{f,p} - 9\mathtt{ndof}_{g,p} + 7)$
	HDG	$3n_{e,p}\,\mathtt{ndof}_{f,p}$
	CG(NSC)	$n_{e,p}(\mathtt{ndof}_{e,p} - 3\mathtt{ndof}_{f,p} + 3\mathtt{ndof}_{g,p} - 1)$
	CDG	$n_{e,p}\,\mathtt{ndof}_{e,p}$

Table II. Expressions for `nnz` for different methods.

		nnz
Triangles	CG	$n_{e,p}(15\mathtt{ndof}^2_{f,p} - 36\mathtt{ndof}_{f,p} + 19)/2$
	HDG	$15n_{e,p}\,\mathtt{ndof}^2_{f,p}/2$
	CG(NSC)	$n_{e,p}(2\mathtt{ndof}^2_{e,p} - 3\mathtt{ndof}^2_{f,p} + 1)/2$
	CDG	$n_{e,p}\,\mathtt{ndof}_{e,p}(\mathtt{ndof}_{e,p} + 3\mathtt{ndof}_{f,p})$
Quads	CG	$n_{e,p}(14\mathtt{ndof}^2_{f,p} - 32\mathtt{ndof}_{f,p} + 17)$
	HDG	$14n_{e,p}\,\mathtt{ndof}^2_{f,p}$
	CG(NSC)	$n_{e,p}(\mathtt{ndof}^2_{e,p} - 2\mathtt{ndof}^2_{f,p} + 1)$
	CDG	$n_{e,p}\,\mathtt{ndof}_{e,p}(\mathtt{ndof}_{e,p} + 4\mathtt{ndof}_{f,p})$
Tets	CG	$n_{e,p}(84\mathtt{ndof}^2_{f,p} - 288\mathtt{ndof}_{f,p}\,\mathtt{ndof}_{g,p} +$ $+223\mathtt{ndof}^2_{g,p} + 192\mathtt{ndof}_{f,p} - 288\mathtt{ndof}_{g,p} + 95)/6$
	HDG	$14n_{e,p}\,\mathtt{ndof}^2_{f,p}$
	CG(NSC)	$n_{e,p}(6\mathtt{ndof}^2_{e,p} - 12\mathtt{ndof}^2_{f,p} + 7\mathtt{ndof}^2_{g,p} - 1)/6$
	CDG	$n_{e,p}\,\mathtt{ndof}_{e,p}(\mathtt{ndof}_{e,p} + 4\mathtt{ndof}_{f,p})$
Hexes	CG	$3n_{e,p}(11\mathtt{ndof}^2_{f,p} - 48\mathtt{ndof}_{f,p}\,\mathtt{ndof}_{g,p} +$ $+49\mathtt{ndof}^2_{g,p} + 32\mathtt{ndof}_{f,p} - 64\mathtt{ndof}_{g,p} + 21)$
	HDG	$33n_{e,p}\,\mathtt{ndof}^2_{f,p}$
	CG(NSC)	$n_{e,p}(\mathtt{ndof}^2_{e,p} - 3\mathtt{ndof}^2_{f,p} + 3\mathtt{ndof}^2_{g,p} - 1)$
	CDG	$n_{e,p}\,\mathtt{ndof}_{e,p}(\mathtt{ndof}_{e,p} + 6\mathtt{ndof}_{f,p})$

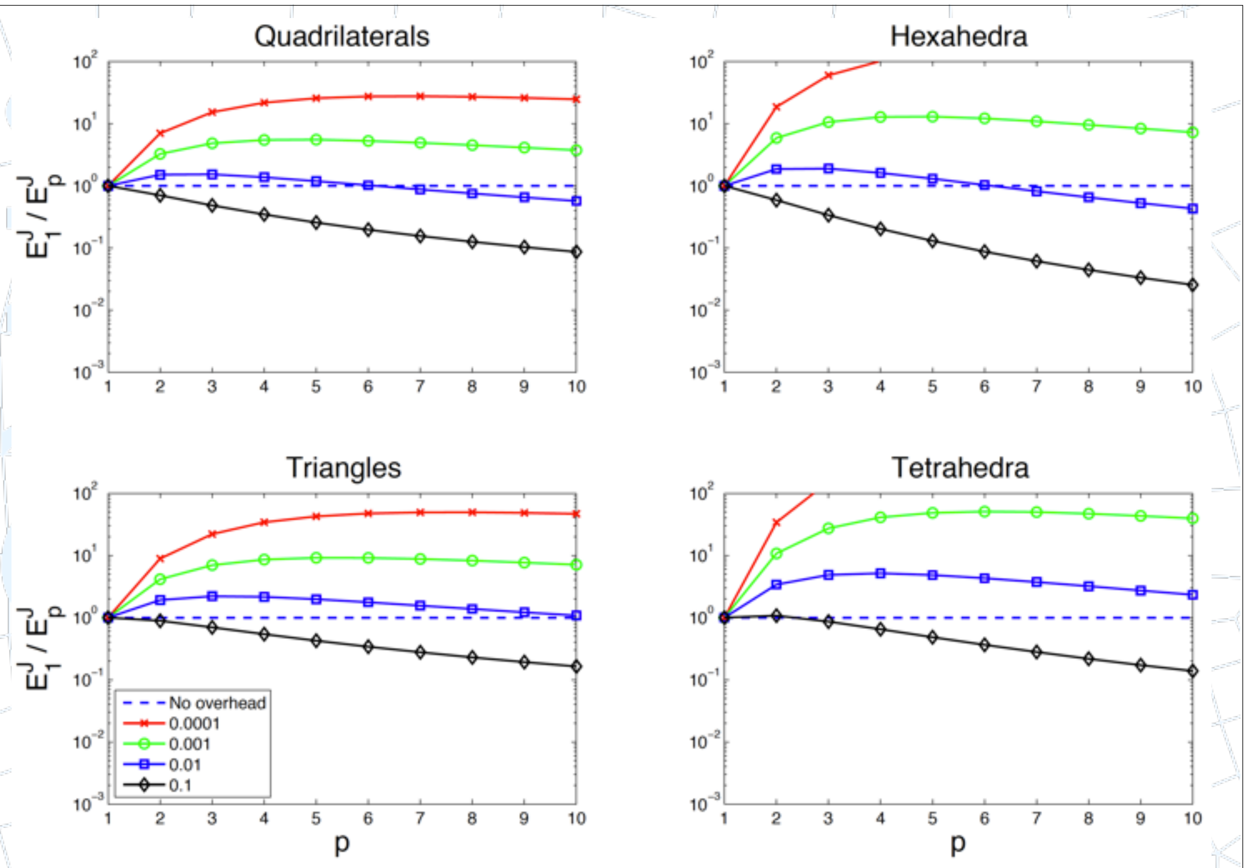


Figure 3. Flops ratio (linear/order p) for creating element matrices of straight-sided elements.

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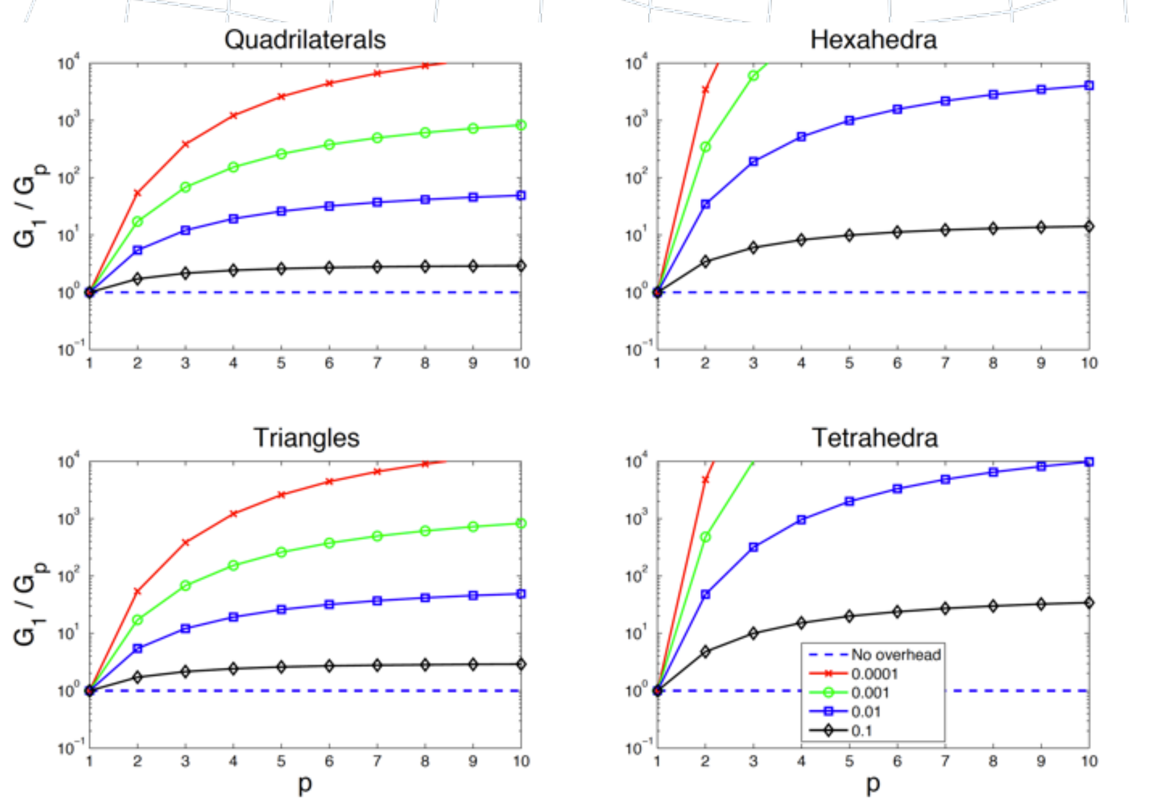

Figure 8. Flops ratio (linear/order p) for the global solve in HDG with a sparse direct solver after nested dissection renumbering.

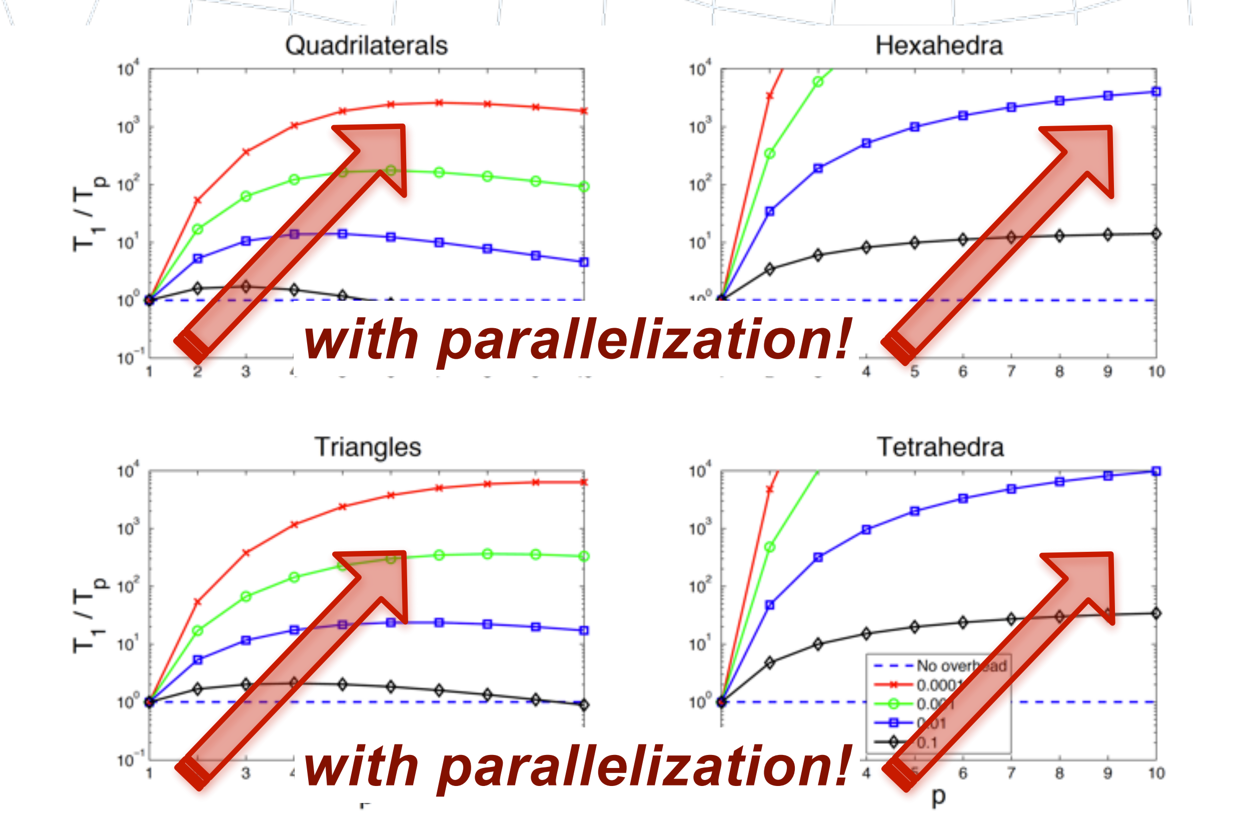

Figure 10. Total Flops ratio (linear/order p) for HDG with a sparse direct solver and worse case scenario for a problem with 100 wavelengths per domain, $k = 100$.

HIGH vs. LOW order

- Based on FLOPS (not asymptotic, not runtime,...)
- High-order approximations outperform low-order (smooth solutions)
  - ✓ in 2D and more in 3D
  - ✓ at engineering accuracy or higher (2 digits)
  - ✓ always for global solves (implicit)
  - ✓ also for element-by-element (explicit) if straight-sided elements or sum-factorization is used
- Only case for p=1: explicit codes and non-linear problems and majority of curved elements

[AH, A. Aleksandar, X. Roca, J. Peraire, IJNME'13]
[G. Giorgiani, D. Modesto, S. Fernandez-Mendez, AH, IJNMF'13]

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Continuous versus Discontinuous

- All the freedom to choose interpolation functions element-by-element, numerical flux stabilization, data structure, local conservation, …
  pays-off the overhead of edge/face node duplication?

- Only for explicit schemes…
- NOT for low-order but YES for high-order approximations...

- Can DG outperform CG in an implicit problem?

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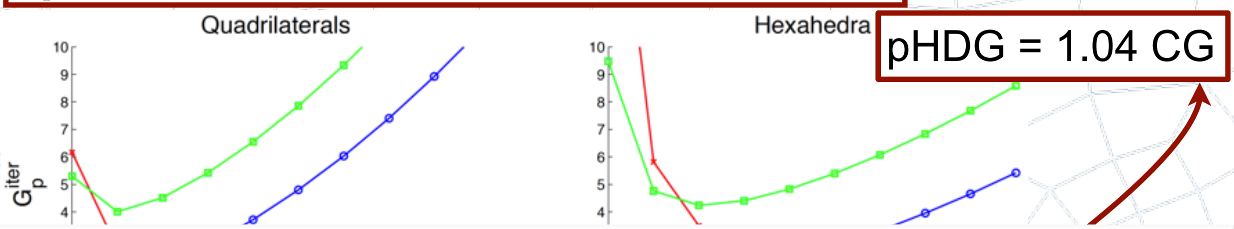
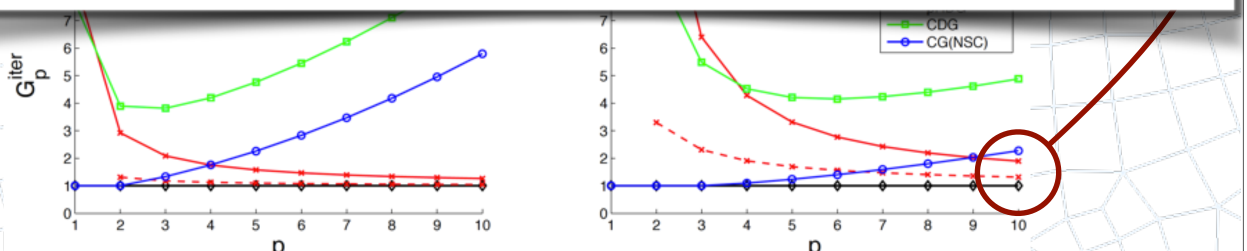
$$\boxed{G_p^{iter} = \mathrm{SpMV} + \mathrm{SpFS} + \mathrm{SpBS} = 4\mathrm{nnz} + \mathrm{ndof}}$$

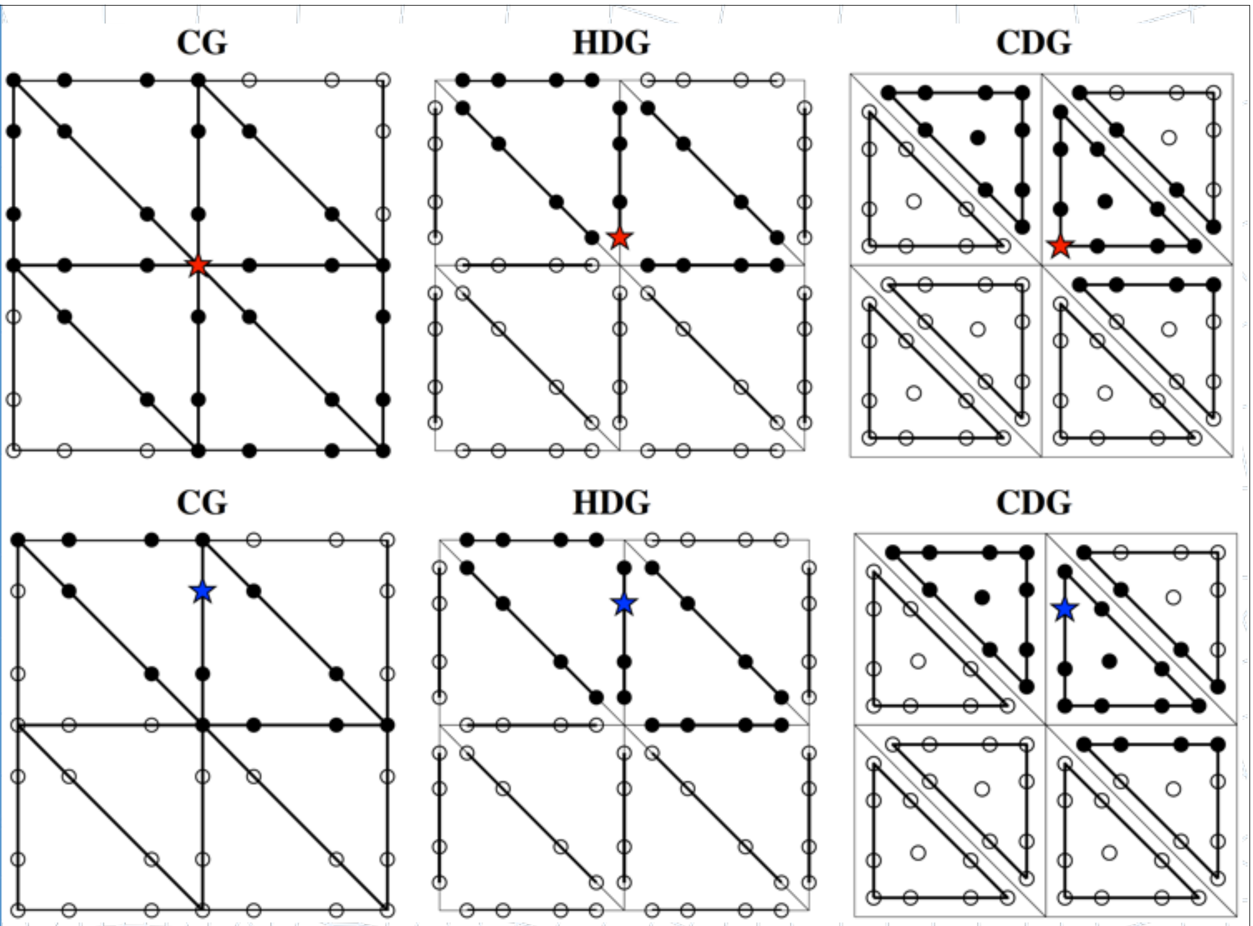


pHDG = 1.04 CG

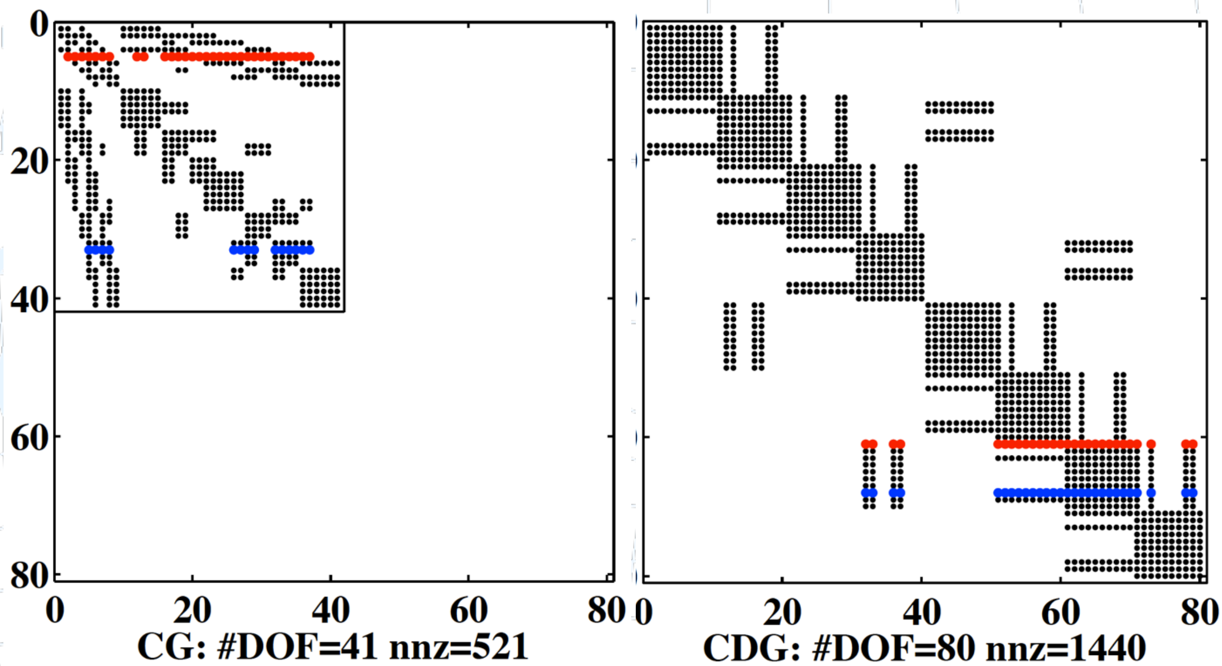
REMEMBER:
Number crunching is NOT our goal
Computability is NOT only operation count



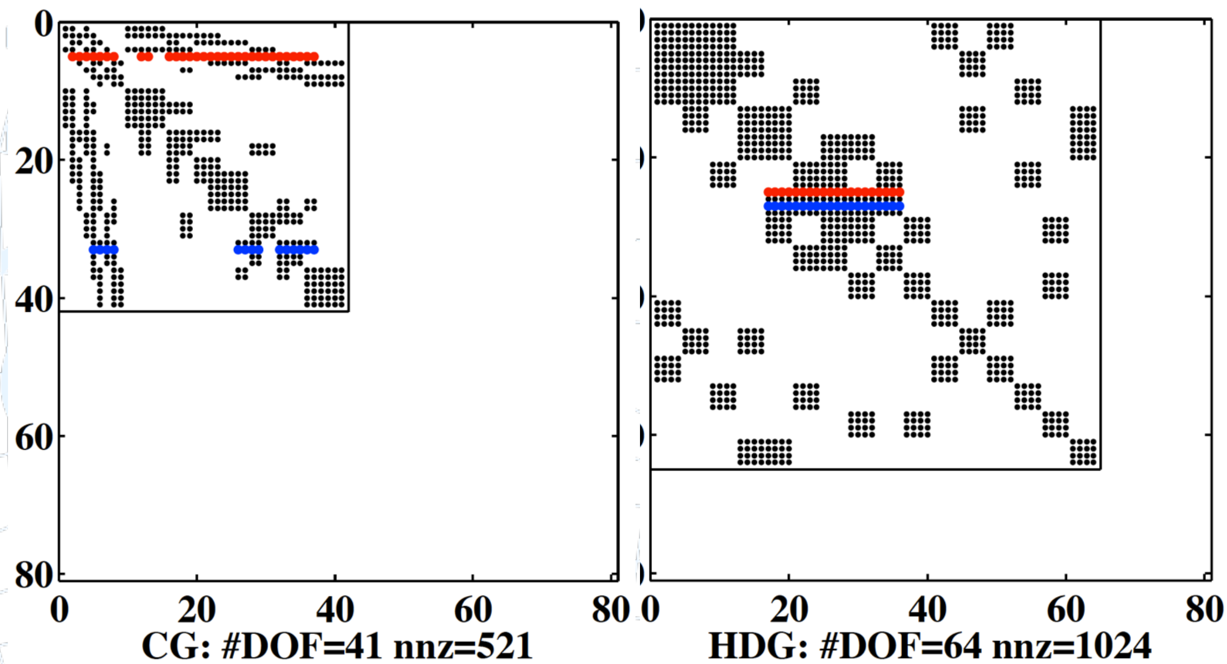
[AH, A. Aleksandar, X. Roca, J. Peraire IJNME'13]



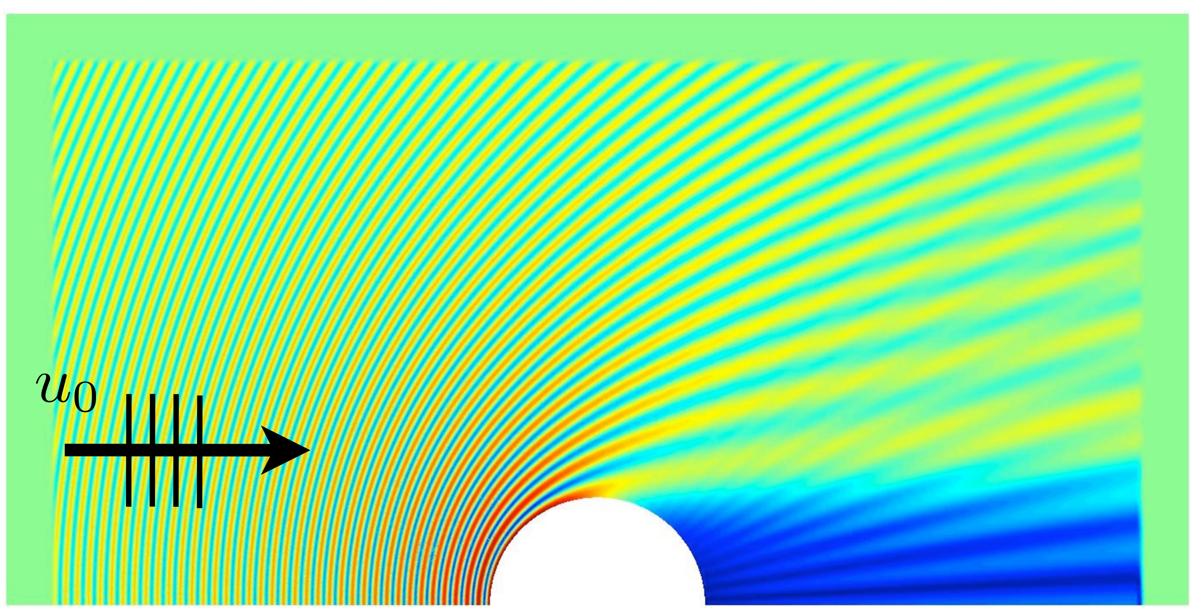
Stencil: CG versus CDG

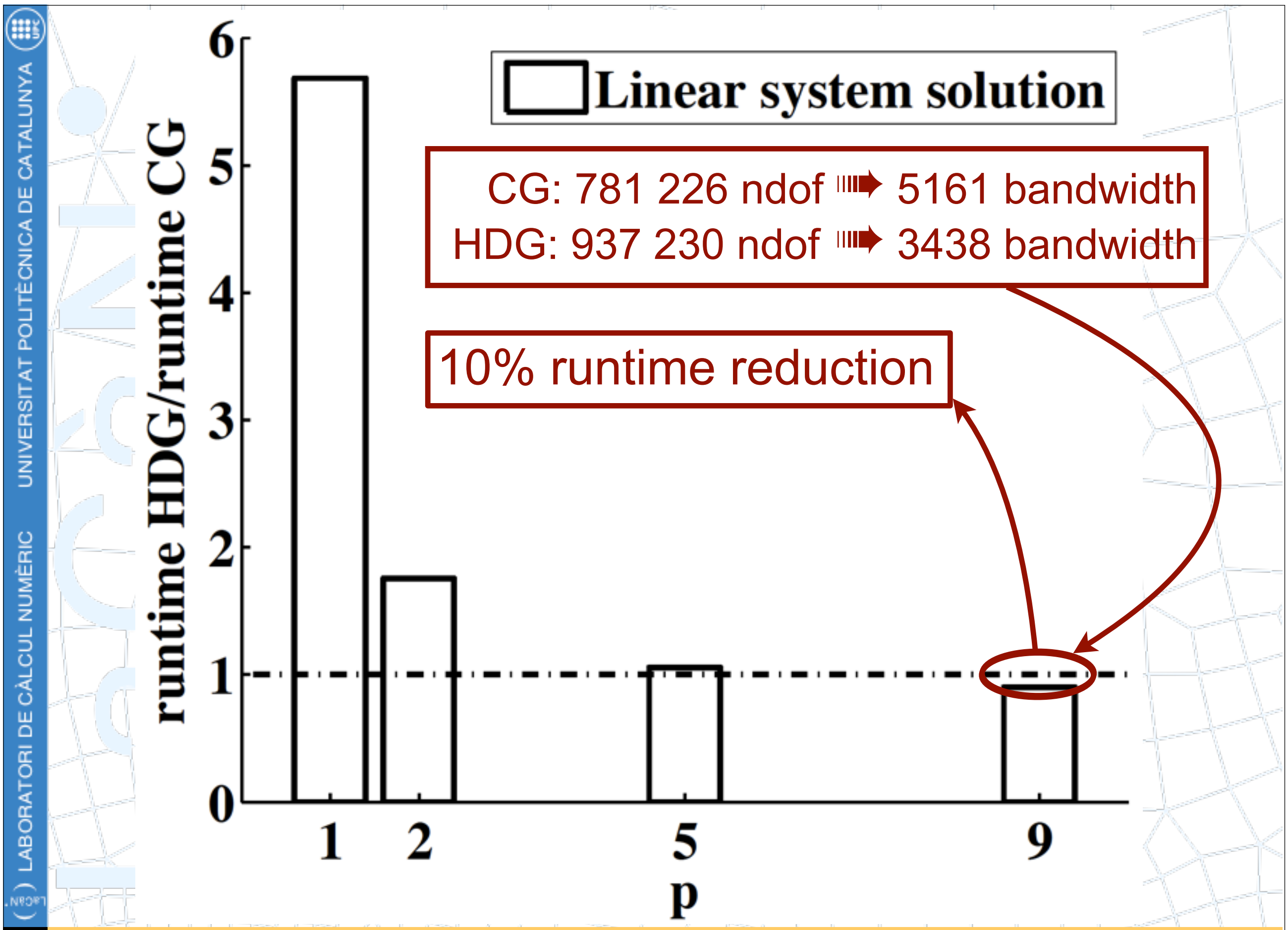


Stencil: CG versus HDG

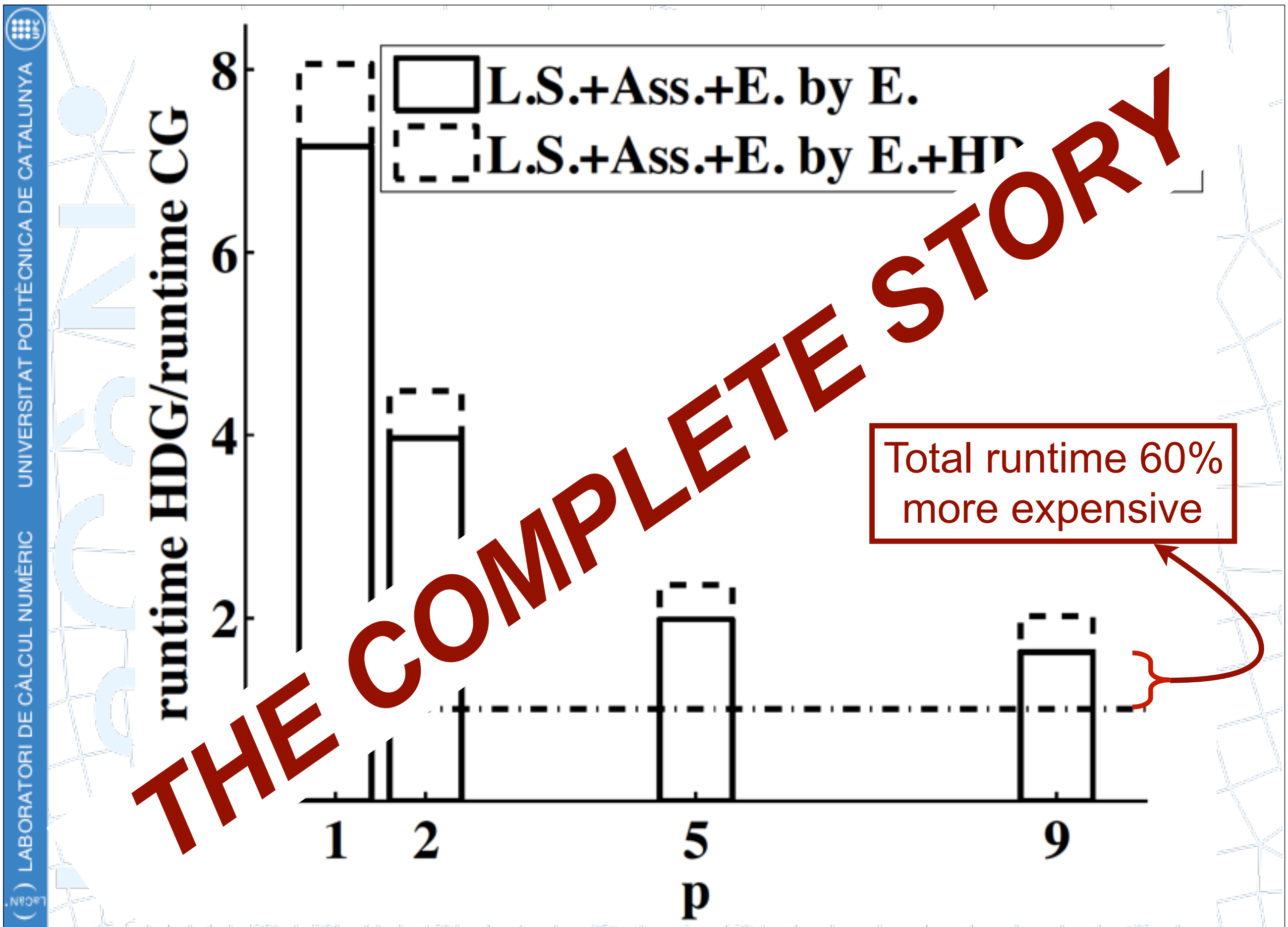


Scattering circle

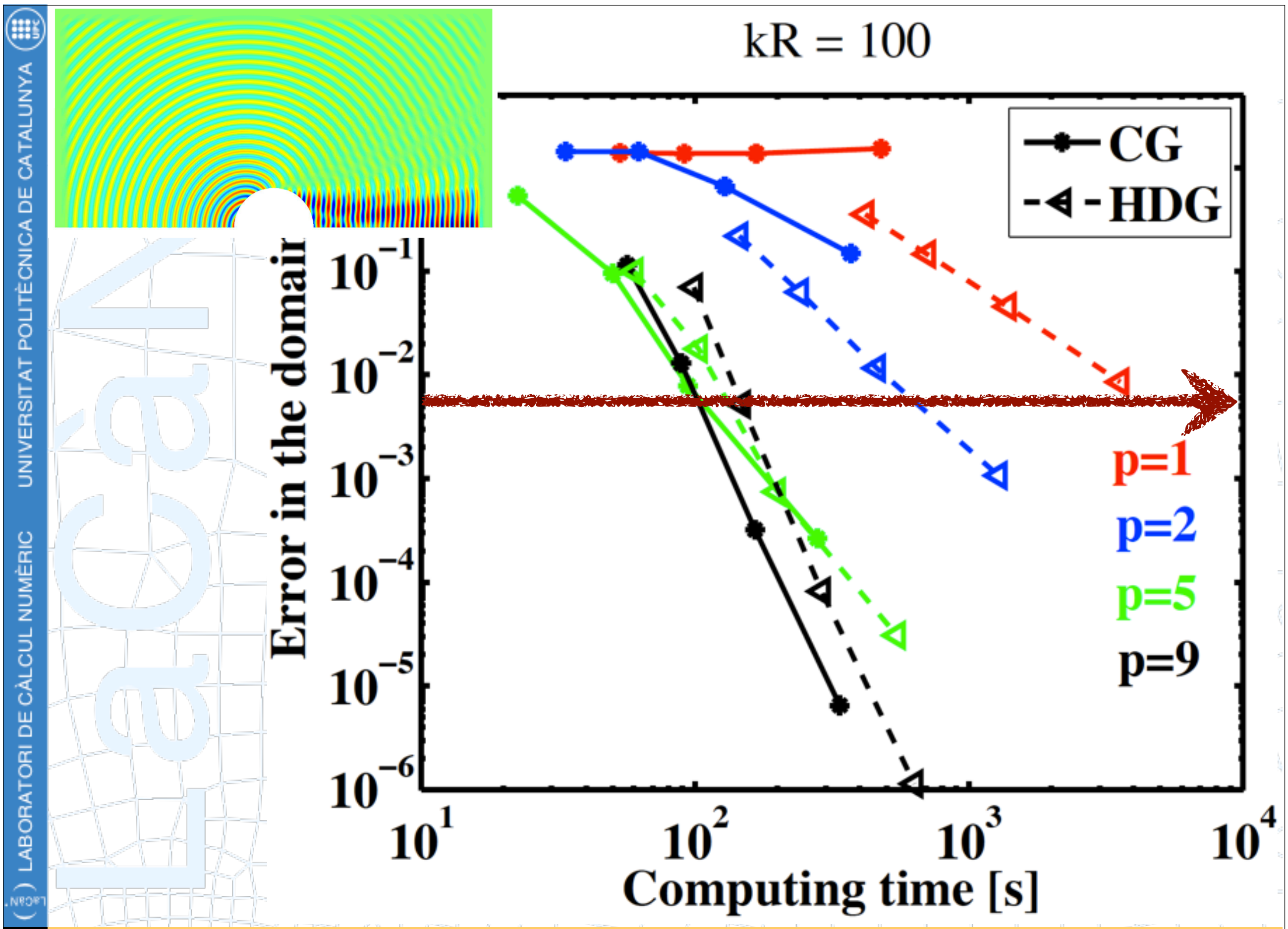




Linear system solution
CG: 781 226 ndof ➡ 5161 bandwidth
HDG: 937 230 ndof ➡ 3438 bandwidth
10% runtime reduction
runtime HDG/runtime CG
0
1
2
3
4
5
6
1
2
5
9
p



L.S.+Ass.+E. by E.
L.S.+Ass.+E. by E.+HP
THE COMPLETE STORY
Total runtime 60% more expensive
runtime HDG/runtime CG
2
4
6
8
1
2
5
9
p


kR = 100
CG
HDG
p=1
p=2
p=5
p=9
Error in the domai
Computing time [s]
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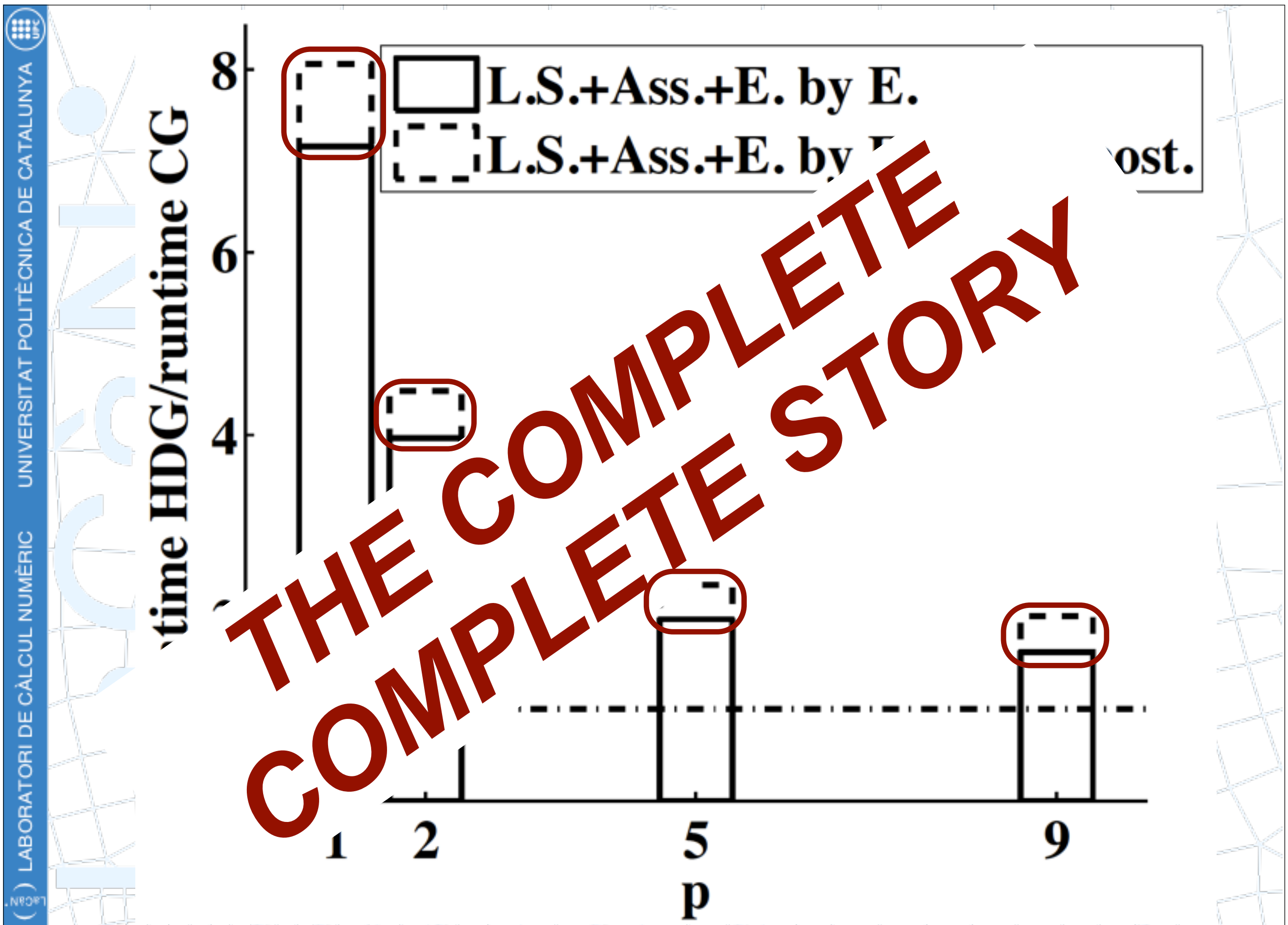
L.S.+Ass.+E. by E.
L.S.+Ass.+E. by
ost.
time HDG/runtime CG
p
THE COMPLETE
COMPLETE STORY
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ADAPTIVITY process

- CHEAP and RELIABLE error estimate

$$E_K^2 = \frac{1}{A_K} \int_K (\boldsymbol{u}^* - \boldsymbol{u})^2 \, d\Omega$$

HDG superconvergent solution

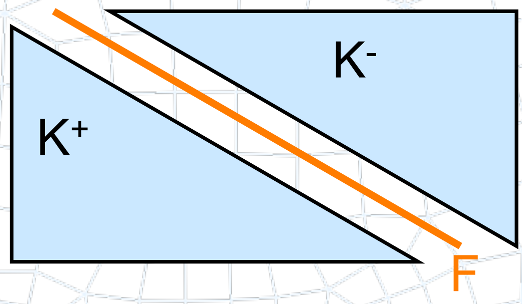
- Degree update in each element K (inspired in [Remacle, Flaherty & Shepard'03])

  Goal: uniform error distribution

$$\Delta p_K = \left\lceil \log_b(E_K/\epsilon_K) \right\rceil$$

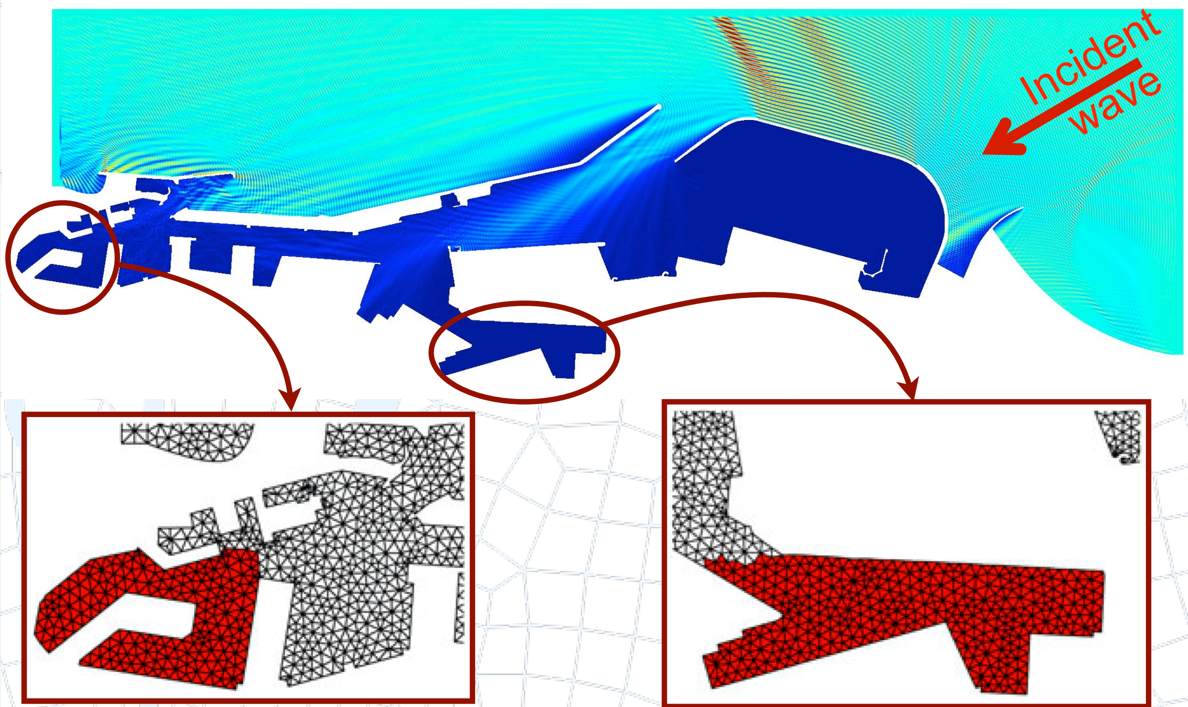
- Degree update for faces [Cockburn, Chen'12]

$$p_F = \max\{p_{K^+}, p_{K^-}\}$$



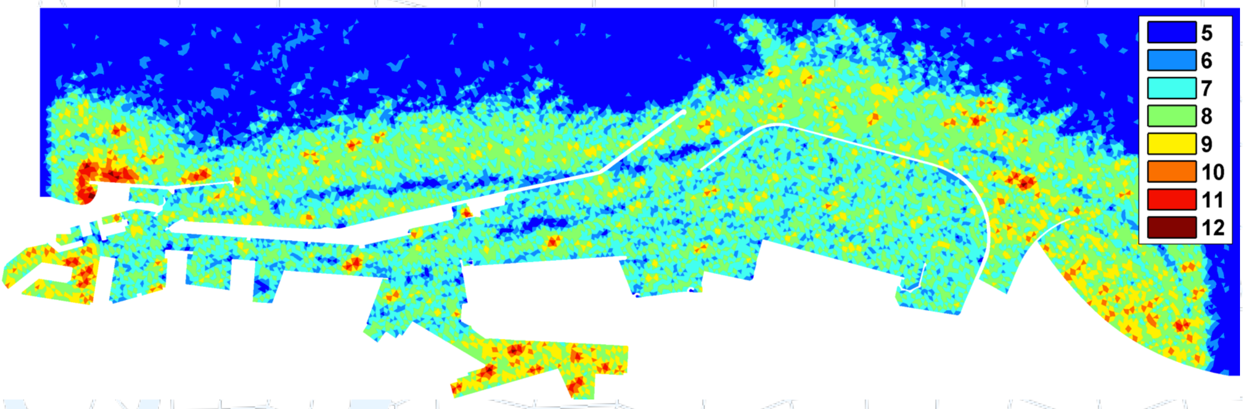
Example: Barcelona harbor

Two areas of interest

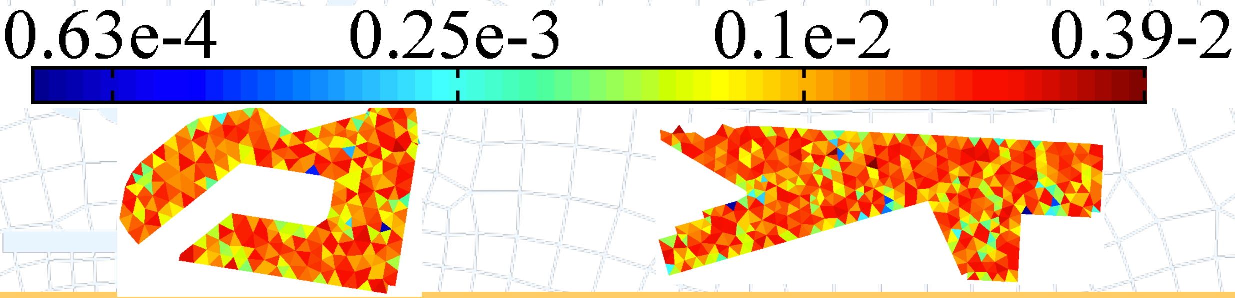


[G. Giorgiani, S. Fernandez-Mendez, AH IJNMF'13]

- Final p-map



- Final error map in the two areas of interest:



Barcelona harbor